Routing on a Ring Network

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Abstract. We study routing on a ring network in which traffic originates from nodes on the ring and is destined to the center. The users can take direct paths from originating nodes to the center and also multihop paths via other nodes. We show that routing games with only one and two hop paths and linear costs are potential games. We give explicit expressions of Nash equilibrium flows for networks with any generic cost function and symmetric loads. We also consider a ring network with random number of users at nodes, all of them having same demand, and linear routing costs. We give explicit characterization of Nash eequilibria for two cases: (i) General i.i.d. loads and one and two hop paths, (ii) Bernoulli distributed loads. We also analyze optimal routing in each of these cases.

Keywords: ring network, routing games, potential games, Nash equilibrium, optimal flow configuration

1 Introduction

Routing problems arise in networks in which common resources are shared by a group of users. Examples of such scenario include flow routing in communication networks, traffic routing in transportation networks, flow of work in manufacturing plants etc. Each user incurs a certain cost (e.g., delay) at each link on its route, where the cost depends on the flows through the link. The routing problems, when handled by a centralized controller, aim to optimize the aggregate cost of all the users, e.g., average network delay. However, a centralized solution may not be viable for several reasons. For instance, a very large network and its time varying attributes (e.g., traffic and link states in a communication network) could lead to excessive communication overhead for solving the problem centrally. In other cases, the very premise of the network may be such that local administrators control different portions of the network, e.g., different parts of a transportation network may be controlled by different depots. In either case, distributed controllers may compete to maximize individual, and often conflicting, performance measures. It is imperative to assess the performance of distributed control, especially how far it is from the global optimal.

Distributed control of routing has been widely modelled as noncooperative games among self-interested decision makers. Nash equilibria of the games

(Wardrop equilibria in case of nonatomic games) characterize the system-wide flow configuration resulting from such distributed control. Wardrop [10] introduced Wardrop equilibrium in the context of transportation networks, and Dafermos and Sparrow [3] showed that it can be characterized as a solution of a standard network optimization problem. Orda et al [8] showed existence and uniqueness of Nash equilibrium in routing games under various assumptions on the cost function. They also showed a few interesting monotonicity properties of the Nash equilibria. Cominetti et al [2] computed the worst-case inefficiency of Nash equilibria and also provided a pricing mechanism that reduces the worst-case inefficiency. Altman et al [1] considered a class of polynomial link cost functions and showed that these lead to predictable and efficient Nash equilibria. Hanwal et al [4] studied routing over time and studied a stochastic game resulting from random arrival of traffic.

We study a routing problem on a ring network in which users' traffic originate at nodes on the ring and are destined to a common node at the centre (see Fig. 1). Each user can use the direct link from its node to the centre and also a certain number of paths through the adjacent nodes, to transport its traffic. The users incur two costs: (i) The cost of using a link between a node at the ring and the centre, (ii) the cost of redirecting the traffic through adjacent nodes. The number of users attached/connected to the node can be random. We characterize Nash equilibria of such routing games.

Scheduling problems are a class of resource allocation problems in which resources are shared over time. In these problems, unlike simultaneous action routing problems, each user may see the system state that results from its predecessor's actions. However, if we assume that such information is not available to the users, our framework can also be used to analyze certain scheduling (or, temporal routing) problem.

In Section 2, we formally introduce our general framework and also illustrate how it can be used to model several problems arising in communication networks, transportation networks etc. In subsequent sections, we analyze special cases of this framework. Following is a brief outline of our contribution

1. In Section 3, we show that routing games with only one and two hop paths and linear costs are potential games. We also give explicit expressions of Nash equilibrium flows for networks with any generic cost function and symmetric loads.
2. In Section 4 we consider networks with random loads and linear routing costs. We give explicit characterization of Nash eequilibria for two cases: (i) General load distribution and one and two hop paths, (ii) Bernoulli distributed loads.

2 System Model

Let us consider a ring network with N nodes and M_n users at each node $n \in [N] := \{1, \dots, N\}$. Let us assume that the ith user at node n has a flow requirement ϕ_n^i to be sent to the centre. Let $c(z)$ represent the cost per unit of flow at any link where z is the aggregate traffic through this link. Throughout we assume that $c(\cdot)$ is positive, strictly increasing and convex. We assume that each user can use the direct link to the centre and the K other links through K adjacent nodes in the clockwise direction. For example, any user at at node n can use links $(n,0),(n+1,0),\ldots,(n+K,0)$.[3] We also assume that a user at node n incurs kd additional per unit flow cost for any flow that it routes through link $(n+k,0)$. Note that we assume no cost for using the links along the ring.

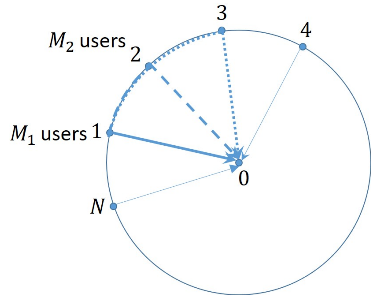

Fig. 1. A ring network with $K=2$. For example, the users at node 1 can use paths $(1,0),(1,2,0)$ and $(1,2,3,0)$.

For each $n \in [N]$, $i \in [M_n]$, $l \in [n,n+K]$, let x^i_{nl} be the flow of ith user at node n that is routed through link l. For brevity, we let x^i_n denote the flow configuration of the ith user at node n, x denote the network flow configuration and x_l denote the total flow through link l; $x^i_n = (x^i_{nl}, l \in [n,n+K]), x = (x^i_n, n \in [N], i \in [M_n])$ and $x_l = \sum_{n\in[l-K,l]}\sum_{i\in[M_n]} x^i_{nl}$. Then the total cost incurred by this user is

$$C^i_n(x) = \sum_{l\in[n,n+K]} x^i_{nl}(c(x_l)+(l-n)d), \tag{1}$$

and the aggregate network cost is $C(x) = \sum_{n\in[N]}\sum_{i\in[M_n]} C^i_n(x)$. Note that the flows must satisfy

$$\sum_{l\in[n,n+K]} x^i_{nl} = \phi^i_n \tag{2}$$

for all $i \in [M_n], n \in [N]$ in addition to nonnegativity constraints.

We now illustrate how this framework can be used to model a variety of routing and scheduling problems.

1. We can think of this framework as modeling routing in a transportation network in a city. The ring and the centre represent a ring road and the city centre, respectively. We have sets of vehicles starting from various entry

[3] Clearly, the addition here is modulo N.

points, represented as nodes on the ring, all destined to the city centre. The costs here represent latency. We assume that the ring road has large enough capacity to render the latency along it independent of the load. On the other hand, latency on the roads joining the ring to the centre is traffic dependent. Each node has a set of depots, each controlling routing of a subset of vehicles starting at this node, and interested in minimizing routing costs of only those vehicles.
2. We can use this framework to model load balancing in distributed computer systems [5].
3. We can also use this framework to model scheduling of charging of electric vehicles at a charging station. Here, the nodes represent time slots and players represent vehicles. The per unit charing cost in a slot depends on the charge drawn in that slot. Each vehicle can wait up to K slots to be charged. Further, each vehicle would like to minimize a weighted sum of its charging cost and waiting time. We assume that the vehicles do not know pending charge from the earlier vehicles when making scheduling decision.

3 Deterministic Loads

Nash Equilibrium: A flow configuration x is a Nash equilibrium if, for all $i \in [M_n], n \in [N]$,

$$C_n^i(x) = \min_{y_n^i} C_n^i(y_n^i, x \setminus x_n^i) \tag{3}$$

subject to (2) and nonnegative constraints. Under our assumptions on $c(\cdot)$, the routing game is a *convex game* [9]. Existence and uniqueness of the Nash equilibrium then follows from [8]. It follows that the equilibrium is characterized by the following Kuhn-Tucker conditions(using cost from equation (1)): for every $i \in [M_n]$ there exists a Lagrange multiplier λ_n^i such that, for every link $l \in [n, n+K]$,

$$c(x_l) + (l-n)d + x_{nl}^i c'(x_l) \geq \lambda_n^i \tag{4}$$

with equality if $x_{nl}^i > 0$. From this,

$$\lambda_n^i = \frac{\sum_{l:x_{nl}^i>0} \frac{c(x_l)+(l-n)d}{c'(x_l)} + \phi_n^i}{\sum_{l:x_{nl}^i>0} \frac{1}{c'(x_l)}}, \text{ for all } i \in [M_n], n \in [N].$$

We observe that the equilibrium flow configuration is the solution of the following system of equations.

$$x_{nj}^i = \max\left\{ \frac{1}{c'(x_j)} \frac{\sum_{l:x_{nl}^i>0} \frac{c(x_l)-c(x_j)+(l-j)d}{c'(x_l)} + \phi_n^i}{\sum_{l:x_{nl}^i>0} \frac{1}{c'(x_l)}}, 0 \right\}, \tag{5}$$

for all $i \in [M_n], j \in [n, n+K], n \in [N]$.

Optimal Solution: Recall that the aggregate routing cost is

$$\sum_{n\in[N]}\left(x_n c(x_n) + d\sum_{l\in[n,n+K]}(l-n)\sum_{i\in[M_n]}x_{nl}^i\right).$$

Clearly, for each $n \in [N]$, we only need to optimize the total requirement of the users $[M_n]$ which flows through link $l \in [n, n+K]$, denoted as x_{nl}. Individual user's flows can always be chosen to match the aggregate optimal flow requirements. We can thus restate the optimal routing problem as

$$\min \sum_{n\in[N]} x_n\left(c(x_n) + d\sum_{l\in[n,n+K]}(l-n)x_{nl}\right),$$

$$\text{subject to} \sum_{l\in[n,n+K]} x_{nl} = \phi_n := \sum_{i\in[M_n]}\phi_n^i, \text{for all } n \in [N],$$

and nonnegativity constraints. The optimal flows satisfy the following Kuhn-Tucker conditions: for every $n \in [N]$ there exist a Lagrange multiplier λ_n such that, for every link $l \in [n, n+K]$,

$$c(x_l) + (l-n)d + x_l c'(x_l) \geq \lambda_n$$

with equality if $x_{nl} > 0$. From this,

$$\lambda_n = \frac{\sum_{l:x_{nl}>0}\left(\frac{c(x_l)+(l-n)d}{c'(x_l)} + x_{-nl}\right) + \phi_n}{\sum_{l:x_{nl}^i>0}\frac{1}{c'(x_l)}}, \text{ for all } n \in [N],$$

where x_{-nl} is the total flow through link $(l, 0)$ that has not originated from node n. We observe that the optimal flow configuration is the solution of the following system of equations.

$$x_{nj} = \max\left\{\frac{1}{c'(x_j)}\frac{\sum_{l:x_{nl}>0}\left(\frac{c(x_l)-c(x_j)+(l-j)d}{c'(x_l)} + x_{-nl} - \frac{c'(x_j)x_{-nj}}{c'(x_l)}\right)+\phi_n}{\sum_{l:x_{nl}^i>0}\frac{1}{c'(x_l)}}, 0\right\}, \quad (6)$$

for all $j \in [n, n+k], n \in [N]$.

We can elegantly obtain Nash equilibria and optimal solutions in special cases. In the following we consider two such cases, the first allowing only one-hop and two-hop paths to the centre and linear costs, and the second having same number of users, all with identical requirements, at all the nodes.

3.1 Maximum Two Hops and Linear Costs ($K = 1$ and $c(x) = x$)

Here, using $x_{nn}^i + x_{n(n+1)}^i = \phi_n^i$, from (5),

$$x_{nn}^i = \left[\frac{c(x_{n+1}) - c(x_n) + d + c'(x_{n+1})\phi_n^i}{c'(x_n) + c'(x_{n+1})}\right]_0^{\phi_n^i}$$

for all $i \in [M_n], n \in [N]$.[4] For linear costs, substituting $c(x) = x$ for all x,

$$x^i_{nn} = \left[\frac{x_{n+1} - x_n + d + \phi^i_n}{2}\right]_0^{\phi^i_n}. \quad (7)$$

Further, using $x_n = \sum_{j\in[M_n]} x^j_{nn} + \sum_{j\in[M_{n-1}]}(\phi^j_{n-1} - x^j_{(n-1)(n-1)})$ and $x_{n+1} = \sum_{j\in[M_n]}(\phi^j_n - x^j_{nn}) + \sum_{j\in[M_{n+1}]} x^j_{(n+1)(n+1)}$,

$$x^i_{nn} = \left[\frac{2\phi^i_n + \sum_{j\in M_n\setminus i}(\phi^j_n - 2x^j_{nn})}{4} + \frac{\sum_{j\in M_{n+1}} x^j_{(n+1)(n+1)} - \sum_{j\in M_{n-1}}(\phi^j_{n-1} - x^j_{(n-1)(n-1)}) + d}{4}\right]_0^{\phi^i_n}. \quad (8)$$

Notice that flow configuration of ith user at node n is completely specified by x^i_{nn}. The above equation can be seen as the best response of this user.

Lemma 1. *If the players update according to*(8)*, round-robin or random update processes converge to the Nash equilibrium.*

Proof. The routing game under consideration is a potential game with potential function

$$V(x) = \frac{1}{2}\left(\sum_{n\in[N]}\left(x_n^2 + \sum_{i\in[M_n]}((x^i_{nn})^2 + (x^i_{n(n+1)})^2 + 2x^i_{n(n+1)}d)\right)\right).$$

Hence it exhibits the improvement property and convergence as stated in the lemma [7].

Optimal Solution: From (6),

$$x_{nn} = \left[\frac{c(x_{n+1}) - c(x_n) + d + c'(x_{n+1})(x_{n+1} - x_{n(n+1)} + \phi_n) - c'(x_n)(x_n - x_{nn})}{c'(x_n) + c'(x_{n+1})}\right]_0^{\phi_n}$$

for all $n \in [N]$. For linear costs, substituting $c(x) = x$ for all x,

$$x_{nn} = \left[\frac{2x_{n+1} - 2x_n - x_{n(n+1)} + x_{nn} + d + \phi_n}{2}\right]_0^{\phi_n}.$$

Further, using $x_n = x_{nn} + \phi_{n-1} - x_{(n-1)(n-1)}$, $x_{n+1} = \phi_n - x_{nn} + x_{(n+1)(n+1)}$ and $x_{n(n+1)} = \phi_n - x_{nn}$,

$$x_{nn} = \left[\frac{x_{(n-1)(n-1)} + x_{(n+1)(n+1)}}{2} + \frac{2(\phi_n - \phi_{n-1}) + d}{4}\right]_0^{\phi^i_n}. \quad (9)$$

[4] $[x]^b_a := \min\{\max\{x, a\}, b\}$.

Notice that the aggregate flow configuration of the users at node n is completely specified by x_{nn}. As for the game problem if we update according to(9), round-robin or random update processes converge to the optimal solution.

In fact, we can provide a more explicit characterization of the Nash equilibrium when $M_n = M$ for all $n \in [N]$ and $x^i_{nl} > 0$ for all $i \in [M], l \in \{n, n+1\}$ and $n \in [N]$. Also, we can show that, when $M_n = M$, there cannot be an optimal solution with $x_{nl} > 0$ for all $l \in \{n, n+1\}$ and $n \in [N]$. This analysis is along the lines of [6] and is presented in Appendix A.

3.2 Symmetric Loads ($M_n = M$ and $\phi^i_n = \phi$)

Here, we can restrict to symmetric flow configurations owing to symmetry of the problem. We can express any symmetric network flow configuration as a vector $\beta = (\beta_0, \beta_1 \ldots, \beta_K)$, $\sum_{j=0}^{K} \beta_j = \phi$ where $\beta_j := x^i_{n(n+j)}$ for all $i \in [M], n \in [N]$ and $j \in [0, K]$.

Theorem 1. *If $M_n = M$ and $\phi^i_n = \phi$ for all $i \in [M_n], n \in [N]$, then the unique Nash equilibrium is*

$$\beta_j = \begin{cases} \frac{\phi}{k^*+1} + \frac{(K^*-2j)d}{2c'(M\phi)} & \text{if } l \in [n, n+K^*] \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

where $K^* = \min\{\max\{k : k(k+1) < \frac{2\phi c'(M\phi)}{d}\}, K\}$

Proof. From the Karush-Kuhn-Tucker conditions for optimality of β (see (4)),

$$c(\beta_j + x^{-i}_{n(n+j)}) + jd + \beta_j c'(\beta_j + x^{-i}_{n(n+j)}) \geq \lambda \tag{11}$$

where $x^{-i}_{n(n+j)}$ is the total flow on link $(n+j, 0)$ except that of ith user at node n. Note that, for β to be a symmetric Nash equilibrium, $\beta_j + x^{-i}_{n(n+j)} = M\phi$. Hence (11) can be reduced to

$$c(M\phi) + jd + \beta_j c'(M\phi) \geq \lambda$$

with equality if $\beta_j > 0$. So, we see that

$$\beta_j = \max\left\{\frac{1}{c'(M\phi)}(\lambda - c(M\phi) - jd), 0\right\} \tag{12}$$

Note that β_j is decreasing in j. Let us assume that $\beta_k > 0$ for all $k \leq K'$ for some $K' \leq K$, and 0 otherwise. Then, using $\sum_{i=0}^{K'} \beta_i = \phi$ in (12),

$$\lambda(K') - c(M\phi) = \frac{\phi c'(M\phi)}{(K'+1)} + \frac{K'd}{2}, \tag{13}$$

where we write $\lambda(K')$ to indicate dependence of λ on K'. Substituting the above expression in back in (12),

$$\beta_j = \frac{\phi}{1+K'} + \frac{d(K'-2j)}{2c'(M\phi)}, j \in [0, K']. \tag{14}$$

To complete the proof, we claim that K' equals K^* where

$$K^* = \min\{\max\{k : k(k+1) < \frac{2\phi c'(M\phi)}{d}\}, K\}.$$

Let us first argue that K' cannot exceed K^*. We only need to consider the case when $K^* < K$. In this case, from the definition of K^*, for any $K' > K^*$,

$$\frac{1}{K'+1} - \frac{dK'}{2\phi c'(M\phi)} \leq 0,$$

which contradicts the defining property of K' that $\beta_k > 0$ for all $k \leq K'$. This completes the argument. Now we argue that K' cannot be smaller than K^*, again by contradiction. Let $K' < K^*$. Then, from (13),

$$\begin{aligned}\lambda(K') - \lambda(K^*) &= \frac{\phi c'(M\phi)(K^* - K')}{(K^*+1)(K'+1)} - \frac{(K^* - K')d}{2} \\ &= \phi c'(M\phi)(K^* - K')\left\{\frac{1}{(K^*+1)(K'+1)} - \frac{d}{2\phi c'(M\phi)}\right\} > 0,\end{aligned}$$

where the inequality follows from definition of K^*. Hence, from (12),

$$\begin{aligned}\beta_{K^*} &\geq \frac{1}{c'(M\phi)}(\lambda(K') - c(M\phi) - K^*d) \\ &> \frac{1}{c'(M\phi)}(\lambda(K^*) - c(M\phi) - K^*d) > 0.\end{aligned}$$

This contradicts $K' < K^*$ which would imply $\beta_{K^*} = 0$.

Optimal Routing: Observe that the optimal strategy of any user will be $\beta_0 = \phi$ and $\beta_j = 0, 1 \leq j \leq K$.

4 Random Loads

We now consider the scenario where the numbers of users at various nodes, M_n, are i.i.d random variables with distribution $(p_1, p_2 \ldots, p_M)$. Such a case may be used to depict random arrival of electric vehicles at the charging station. We assume that a user knows the number of collocated users but only knows the distribution of users at the other nodes. Throughout this section we restrict to equal flow requirements for all the users, i.e. $\phi_n^i = \phi$ for all $n \in [N], i \in [M_n]$, and linear per unit flow cost, i.e., $c(x) = x$. In the following we analyze two special case of this routing problem, the first assuming the users can only use one-hop and two-hop paths to centre, and the second having Bernoulli user distribution.

4.1 Maximum Two Hops ($K = 1$)

We consider symmetric flow configurations where all the users with equal number of collocated users adopt same flow configuration. We can then express the network flow configuration as a vector $\gamma = (\gamma(1), \gamma(2), \dots)$ where $\gamma(m)$ represents the flow that a user with m collocated users redirects to its two-hop path. Let us define

$$\bar{P}_m = 1 - \sum_{l=m+1}^{M} \frac{lp_l}{l+1}$$
$$\text{and } Q_m = \sum_{l=0}^{m} lp_l,$$

for all $0 \leq m \leq M$.

Theorem 2. *The unique Nash equilibrium is given by*

$$\gamma(m) = \begin{cases} 0, & \textit{if } 1 \leq m \leq m_\alpha \\ \frac{\phi}{2} - \frac{(d-\alpha)}{2(m+1)}, & \textit{otherwise,} \end{cases}$$
$$\textit{where } m_\alpha = \min\{\min\{m : \frac{d}{\phi \bar{P}_m} + \frac{Q_m}{\bar{P}_m} < m+2\}, M\}$$
$$\textit{and } \alpha = d - \frac{d}{\bar{P}_{m_\alpha}} - \frac{Q_{m_\alpha}\phi}{\bar{P}_{m_\alpha}}.$$

Proof. Let us consider a user i with m collocated users. Let us fix the strategies of all other users in the network to $\gamma = (\gamma(1), \gamma(2), \dots)$. Then the best response of user i, say $\gamma'(m)$, is the unique minimizer of the cost function

$$(\phi - \gamma'(m))((m-1)(\phi - \gamma(m)) + \phi - \gamma'(m) + \sum lp_l\gamma(l))$$
$$+ \gamma'(m)((m-1)\gamma(m) + \gamma'(m) + \sum_l lp_l(\phi - \gamma(l)) + d).$$

$\gamma'(m)$ must satisfy the following optimality criterion

$$- 2(\phi - \gamma'(m)) - (m-1)(\phi - \gamma(m)) - \sum lp_l\gamma(l)$$
$$+ 2\gamma'(m) + (m-1)\gamma(m) + \sum lp_l(\phi - \gamma(l)) + d \geq 0$$

with equality if $\gamma'(m) > 0$. For γ to be a symmetric Nash equilibrium, setting $\gamma'(m) = \gamma(m)$ in the above inequality,

$$-(m+1)\phi + 2(m+1)\gamma(m) + \sum lp_l\phi - 2\sum lp_l\gamma(l) + d \geq 0$$

yielding

$$\gamma(m) = \max\{\frac{\phi}{2} + \frac{2\sum lp_l\gamma(l) - \phi\sum lp_l - d}{2(m+1)}, 0\}$$

Clearly, the above should hold for all $m \in \{0, 1, \ldots, M\}$. Setting,

$$\alpha = 2\sum lp_l\gamma(l) - \phi\sum lp_l, \tag{15}$$

$$\text{and } m_\alpha = \lfloor \frac{d-\alpha}{\phi} - 1 \rfloor, \tag{16}$$

we get

$$\gamma(m) = \begin{cases} \frac{\phi}{2} + \frac{\alpha-d}{2(m+1)}, & \text{if } m > m_\alpha \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

We now show how to obtain α and m_α. From (17),

$$mp_m(2\gamma(m) - \phi) = \begin{cases} \frac{(\alpha-d)mp_m}{m+1}, & \text{if } m > m_\alpha \\ -mp_m\phi, & \text{otherwise.} \end{cases}$$

Using this in (15),

$$\begin{aligned} \alpha &= \frac{-d\sum_{m>m_\alpha}\frac{mp_m}{m+1} - \phi\sum_{m=0}^{m_\alpha} mp_m}{(1 - \sum_{m>m_\alpha}\frac{mp_m}{m+1})} \\ &= d - \frac{d}{\bar{P}_{m_\alpha}} - \frac{Q_{m_\alpha}\phi}{\bar{P}_{m_\alpha}}, \end{aligned}$$

and hence, from (16),

$$m_\alpha = \lfloor \frac{d}{\phi\bar{P}_{m_\alpha}} + \frac{Q_{m_\alpha}}{\bar{P}_{m_\alpha}} - 1 \rfloor$$

Let us now turn to the expression of m_α in the statement of the theorem. Clearly, $m_\alpha > \frac{d}{\phi\bar{P}_{m_\alpha}} + \frac{Q_{m_\alpha}}{\bar{P}_{m_\alpha}} - 2$. Also,

$$\frac{d}{\phi\bar{P}_{m_\alpha-1}} + \frac{Q_{m_\alpha-1}}{\bar{P}_{m_\alpha-1}} \geq m_\alpha + 1,$$

implying

$$\begin{aligned} &\frac{d}{\phi\bar{P}_{m_\alpha}} + \frac{Q_{m_\alpha}}{\bar{P}_{m_\alpha}} \geq m_\alpha + 1, \\ \text{or, } &\frac{d}{\phi\bar{P}_{m_\alpha}} + \frac{Q_{m_\alpha}}{\bar{P}_{m_\alpha}} - 1 \geq m_\alpha. \end{aligned}$$

So, the two expressions of m_α are equivalent, and $\gamma(m)$s in the statement of the theorem indeed constitute a Nash equilibrium. Also note that existence of an optimal γ ensures existence of at least one (α, m_α) pair satisfying (15)-(16). It remains to establish uniqueness of (α, m_α) pair satisfying (15)-(16). We do this in Appendix B.

Optimal Routing: The expected total routing cost will be N times the sum of expected routing costs on links $(n-1,n)$ and $(n,0)$ for an arbitrary n. In the following, we optimize the latter to get the optimal flow configuration.

Theorem 3. *The unique optimal flow configuration is given by*

$$\gamma(m) = \begin{cases} 0, & \text{if } 0 \le m \le m_{\bar{\alpha}} \\ \frac{\phi}{2} - \frac{(d-\bar{\alpha})}{4m}, & \text{otherwise,} \end{cases}$$
$$\text{where } m_{\bar{\alpha}} = \min\{\min\{m : \frac{d}{2\phi P_m} + \frac{Q_m}{P_m} < m+1\}, M\}$$
$$\text{and } \bar{\alpha} = d - \frac{d}{P_{m_{\bar{\alpha}}}} - \frac{2Q_{m_{\bar{\alpha}}}\phi}{P_{m_{\bar{\alpha}}}}.$$

Proof. The optimal flow configuration γ is the unique minimizer of the following cost.

$$\sum_{m_n, m_{n-1}} p_{m_n} p_{m_{n-1}} \{(m_{n-1}\gamma(m_{n-1}) + m_n(\phi - \gamma(m_n)))^2 + m_{n-1}\gamma(m_{n-1})d\}$$

For all $0 \le m \le M$, $\gamma(m)$ must satisfy the following optimality criterion.

$$\sum_{l \ne m} 2p_l(2m\gamma(m) - 2l\gamma(l) + (l-m)\phi + \frac{d}{2}) + p_m d \ge 0,$$

with equality if $\gamma(m) > 0$. Equivalently,

$$4m\gamma(m) - 2m\phi - 2\sum p_l(2l\gamma(l) - l\phi) + d \ge 0$$

yielding

$$\gamma(m) = \max\{\frac{\phi}{2} + \frac{4\sum lp_l\gamma(l) - 2\phi\sum lp_l - d}{4m}, 0\}.$$

Setting

$$\bar{\alpha} = 2\sum p_l l(2\gamma(l) - \phi),$$
$$\text{and } m_{\bar{\alpha}} = \lfloor \frac{d-\alpha}{2\phi} \rfloor,$$

we get

$$\gamma(m) = \begin{cases} \frac{\phi}{2} + \frac{\bar{\alpha}-d}{4m}, & \text{if } m > m_\alpha \\ 0, & \text{otherwise.} \end{cases}$$

The remaining steps are similar to those in the proof of Theorem 2 and are omitted for brevity.

Distribution		$\gamma(1)$	$\gamma(2)$	$\gamma(3)$	$\gamma(4)$	$\gamma(5)$
Uniform{0,5}	Optimal	0	0	0	0.125	0.2
	Game	0	0.106	0.203	0.263	0.275
Binomial(5,0.5)	Optimal	0	0	0	0.0821	0.166
	Game	0	0.117	0.2125	0.27	0.308

Table 1:Optimal and game solutions in various distributions for $\phi = 1, d = 2, M = 5, K = 1$.

4.2 Bernoulli Loads ($p_0 + p_1 = 1$)

We again focus on only symmetric flow configuration. As in Section 3.2, we let $x^i_{nl} = \beta_{n-l}$ for all $i \in [M], n \in [N]$ and $l \in [n, n+K]$.

Theorem 4. *The unique Nash equilibrium is given by*

$$\beta_i = \begin{cases} \frac{\phi}{K^*+1} + \frac{d(K^*-2i)}{2(2-p)}, & \text{if } 0 \leq i \leq K^* \\ 0, & \text{otherwise,} \end{cases}$$

where $K^* = \min\{\max\{k : k(k+1) < \frac{2\phi(2-p)}{d}\}, K\}$.

Proof. Let us consider a node i. This node may have a user with probability p, in which we refer to this user as user i. Let all other users in the network have flow configuration $\beta'_j, 0 \leq j \leq K$. Then the following optimization problem yields the best response of user i.

$$\begin{aligned} \min \sum_{j=0}^{j=K} \{\beta_j(\beta_j + jd + \sum_{l \neq i} p\beta'_l)\} \\ \text{subject to} \sum_{j=0}^{K} \beta_j = \phi, \\ \beta_j \geq 0, j \in [0, K]. \end{aligned}$$

From the Karush-Kuhn-Tucker conditions for optimality of β, there exists a Lagrange multiplier λ such that

$$2\beta_j + jd + p \sum_{l \neq j} \beta'_l \geq \lambda, \text{ for all } 0 \leq j \leq K, \tag{18}$$

with equality if $\beta_j > 0$. Note that, for β to be a symmetric Nash equilibrium, $\beta = \beta'$. Hence (18) can be reduced to

$$(2-p)\beta_j + p + jd \geq \lambda.$$

with equality if $\beta_j > 0$. So, we see that

$$\beta_j = \max\left\{\frac{\lambda - p - jd}{2-p}, 0\right\}. \tag{19}$$

The remaining steps are similar to those in the proof of Theorem 1 (see the steps after (12)) and are omitted for brevity.

Optimal Routing: The expected total routing cost will be N times the sum of expected routing costs on links $(n-1, n)$ and $(n, 0)$ for an arbitrary n. In the following, we optimize the latter to get the optimal flow configuration.

Theorem 5. *The unique optimal flow configuration is given by*

$$\beta_j = \begin{cases} \frac{\phi}{K^*+1} + \frac{d(K^*-2j)}{4(1-p)}, & \text{if } 0 \le i \le K^* \\ 0, & \text{otherwise,} \end{cases}$$

where $K^* = \min\{\max\{k : k(k+1) < \frac{4(1-p)\phi}{d}\}, K\}$.

Proof. The optimal routing is the solution of the following optimization problem.

$$\min \sum_{\delta \in \{0,1\}^{K+1}} p^{\sum \delta_j}(1-p)^{(K+1)-\sum \delta_j} \left(\sum_{l=0}^{K} \delta_j \beta_j\right)^2 + pd \sum_{j=0}^{K} j\beta_j$$

$$\text{subject to} \sum_{j=0}^{K} \beta_j = \phi,$$

$$\beta_j \ge 0, \; j \in [0, K].$$

From the Karush-Kuhn-Tucker conditions for optimality of β there exits a Lagrange multiplier λ such that, for all $0 \le j \le K$,

$$2 \sum_{\delta \in \{0,1\}^{K+1}} p^{\sum \delta_l}(1-p)^{(K+1)-\sum \delta_l} \delta_j \left(\sum_{l=0}^{K} \delta_l \beta_l\right) + pjd \ge \lambda,$$

with equality if $\beta_j > 0$. Equivalently,

$$2 \sum_{l=0}^{K} \beta_l \left(\sum_{\delta \in \{0,1\}^{K+1}} p^{\sum \delta_l}(1-p)^{(K+1)-\sum \delta_l} \delta_j \delta_l\right) + pjd \ge \lambda,$$

which reduces to

$$2p\left(\beta_j + p\sum_{l \ne j} \beta_l\right) + pjd \ge \lambda. \tag{20}$$

Using $\beta_j = 1 - \sum_{l \ne j} \beta_l$, (20) further reduces to

$$2p(1-p)\beta_j + 2p^2 + jpd \ge \lambda,$$

with equality if $\beta_j > 0$. So, we see that

$$\beta_j = \max\left\{\frac{\lambda - 2p^2 - jpd}{2p(1-p)}, 0\right\}. \tag{21}$$

The remaining steps are similar to those in the proof of Theorem 1 (see the steps after (12)) and are omitted for brevity.

5 Conclusion and Future Work

We studied routing on a ring network. We studied both, non-cooperative games between competing users and network optimal routing. We considered several special cases of networks with deterministic and random loads. We provided characterization of Nash equilibria and optimal flow configuration in these cases (see Theorems 1-5).

Our future work entails extending this analysis to more general cases. We would like to study price of anarchy, and also pricing mechanisms (tolls) that induce optimality.

Acknowledgments:

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Appendix A

Let us recall the discussion on Nash equilibrium in Section 3.1. When $M_n = M$ for all $n \in [N]$ and $x^i_{nl} > 0$ for all $i \in [M], l \in \{n, n+1\}$ and $n \in [N]$,

$$\begin{aligned} -x_{n+1} + x_n + 2x^i_{nn} &= d + \phi^i_n, \\ \text{and } -x_{n-1} + x_n + 2x^i_{(n-1)n} &= -d + \phi^i_{n-1}. \end{aligned} \tag{22}$$

Adding the above two equations for all $i \in [M]$,

$$-x_{n-1} + 2\left(1 + \frac{1}{M}\right) x_n - x_{n+1} = \frac{1}{M} \sum_{i \in [M]} (\phi^i_{n-1} + \phi^i_n).$$

Further, multiplying the above by u^n and adding over $n \in [N]$,

$$-u(F(u) + x_N - u^N x_N) + 2\left(1 + \frac{1}{M}\right) F(u) - \frac{F(u) - x_1 u + x_1 u^{N+1}}{u} = \psi(u),$$

where

$$F(u) = \sum_{n \in [N]} x_n u^n$$

$$\text{and } \psi(u) = \frac{1}{M} \sum_{n \in [N]} \left(\sum_{i \in [M]} (\phi^i_{n-1} + \phi^i_n) \right) u^n.$$

Rearranging the terms in the above equation

$$-F(u)\left(u^2 - 2\left(1 + \frac{1}{M}\right) u + 1\right) + u^2(1 - u^N)\left(\frac{x_1}{u} - x_N\right) = u\psi(u).$$

Let u_1, u_2 denote the two solutions of $u^2 - 2\left(1 + \frac{1}{M}\right) u + 1 = 0$:

$$u_1 = 1 + \frac{1}{M}(1 - \sqrt{1 + 2M}),\ u_2 = 1 + \frac{1}{M}(1 + \sqrt{1 + 2M}).$$

Substituting $u = u_1, u_2$ in the above equation we obtain the following two linear equations in x_1 and x_N.

$$\begin{aligned} \frac{x_1}{u_1} - x_N &= \frac{\psi(u_1)}{u_1(1 - u_1^N)}. \\ \frac{x_1}{u_2} - x_N &= \frac{\Psi(u_2)}{u_2(1 - u_2^N)}. \end{aligned}$$

These together yield

$$\begin{aligned} x_1 &= \frac{\psi(u_1) u_2 (1 - u_2^N) - \Psi(u_2) u_1 (1 - u_1^N)}{(u_2 - u_1)(1 - u_1^N)(1 - u_2^N)}, \\ x_N &= \frac{\psi(u_1)(1 - u_2^N) - \Psi(u_2)(1 - u_1^N)}{(u_2 - u_1)(1 - u_1^N)(1 - u_2^N)}. \end{aligned}$$

We can now use (22) to obtain the remaining x_ns and (7) to obtain all x^i_{nn}s.

Next, let us focus on optimal routing. We provide via contradiction that an optimal solution cannot have $x_{nl} > 0$ for all $l \in \{n, n+1\}$ and $n \in [N]$. Assume this to be the case. Then, from (9),

$$2x_{nn} - x_{(n-1)(n-1)} - x_{(n+1)(n+1)} = \frac{2(\phi_n - \phi_{n-1}) + d}{2}$$

Adding these for all $n \in [N]$ gives $d = 0$, which contradicts our hypothesis that $d > 0$.

Appendix B

We establish uniqueness via contradiction. Let (α, m_α) and $(\alpha', m_{\alpha'})$ be two pairs satisfying (15)-(16). We assume $m_{\alpha'} > m_\alpha$ without any loss of generality. Recall that $\alpha = d - \frac{d}{\bar{P}_{m_\alpha}} - \frac{\phi Q_{m_\alpha}}{\bar{P}_{m_\alpha}}$ and $\frac{d-\alpha}{\phi} < m_\alpha + 2$, implying

$$\frac{d}{\phi} < (m_\alpha + 2)\bar{P}_{m_\alpha} - Q_{m_\alpha}. \tag{23}$$

Similarly, $\alpha' = d - \frac{d}{\bar{P}_{m_{\alpha'}}} - \frac{\phi Q_{m_{\alpha'}}}{\bar{P}_{m_{\alpha'}}}$ and $\frac{d-\alpha'}{\phi} \geq m_{\alpha'} + 1$, implying

$$\frac{d}{\phi} \geq (m_{\alpha'} + 1)\bar{P}_{m_{\alpha'}} - Q_{m_{\alpha'}}. \tag{24}$$

We argue that $(m_{\alpha'}+1)\bar{P}_{m_{\alpha'}} - Q_{m_{\alpha'}} \geq (m_\alpha+2)\bar{P}_{m_\alpha} - Q_{m_\alpha}$, and hence both (23) and (24) cannot hold simultaneously. Indeed note that

$$\begin{aligned}
&(m_{\alpha'} + 1)\bar{P}_{m_{\alpha'}} - (m_\alpha + 2)\bar{P}_{m_\alpha} \\
&= (m_{\alpha'} + 1)(\bar{P}_{m_{\alpha'}} - \bar{P}_{m_{\alpha'}-1}) + (m_{\alpha'} + 1)\bar{P}_{m_{\alpha'}-1} - (m_\alpha + 2)\bar{P}_{m_\alpha} \\
&= m_{\alpha'} p_{m_{\alpha'}} + \sum_{m=m_\alpha+1}^{m_{\alpha'}-1} \{(m+2)\bar{P}_m - (m+1)\bar{P}_{m-1}\} \\
&\geq m_{\alpha'} p_{m_{\alpha'}} + \sum_{m=m_\alpha+1}^{m_{\alpha'}-1} (m+1)(\bar{P}_m - \bar{P}_{m-1}) \\
&= \sum_{m=m_\alpha+1}^{m_{\alpha'}} m p_m \\
&= Q_{m_{\alpha'}} - Q_{m_\alpha}
\end{aligned}$$

This completes the argument.